Lie symmetry analysis of nonlinear evolution equation for description nonlinear waves in a viscoelastic tube

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Abstract

In this paper, the Lie symmetry method is performed for the nonlinear evolution equation for description nonlinear waves in a viscoelastic tube. we will find one and two-dimensional optimal system of Lie subalgebras. Furthermore, preliminary classification of its group-invariant solutions are investigated.

Keywords. Lie symmetry, Optimal system, Group-invariant solutions, Nonlinear evolution equation.

1 Introduction

The theory of Lie symmetry groups of differential equations was developed by Sophus Lie [10]. Such Lie groups are invertible point transformations of both the dependent and independent variables of the differential equations. The symmetry group methods provide an ultimate arsenal for analysis of differential equations and is of great importance to understand and to construct solutions of differential equations. Several applications of Lie groups in the theory of differential equations were discussed in the literature, the most important ones are: reduction of order of ordinary differential equations, construction of invariant solutions, mapping solutions to other solutions and the detection of linearizing transformations (for many other applications of Lie symmetries see [11], [2] and [1]).

In the present paper, we study the following fifth-order nonlinear evolution equation

$$u_t + auu_x + bu_{x^3} + cu_{x^4} + du_{x^5} = eu_{x^2}. \tag{1}$$

where a,b,c,d and e are positive constants. This equation was introduced recently by Kudryashov and Sinelshchikov [8] which is the generalization of the famous

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Kawahara equation. By using the reductive perturbation method, they obtained the equation (1). Study of nonlinear wave processes in viscoelastic tube is the important problem such tubes similar to large arteries (see [4],[9] and [3]).

In this paper, by using the lie point symmetry method, we will investigate the equation (1) and looking the representation of the obtained symmetry group on its Lie algebra. We will find the preliminary classification of group-invariant solutions and then we can reduce the equation (1) to an ordinary differential equation.

This work is organized as follows. In section 2 we recall some results needed to construct Lie point symmetries of a given system of differential equations. In section 3, we give the general form of a infinitesimal generator admitted by equation (1) and find transformed solutions. Section 4, is devoted to the construction of the group-invariant solutions and its classification which provides in each case reduced forms of equation (1).

2 Method of Lie Symmetries

In this section, we recall the general procedure for determining symmetries for any system of partial differential equations (see [11] and [2]). To begin, let us consider the general case of a nonlinear system of partial differential equations of order nth in p independent and q dependent variables is given as a system of equations

$$\Delta_\nu(x, u^{(n)}) = 0, \qquad \nu = 1, \cdots, l, \tag{2}$$

involving $x = (x^1, \cdots, x^p)$, $u = (u^1, \cdots, u^q)$ and the derivatives of u with respect to x up to n, where $u^{(n)}$ represents all the derivatives of u of all orders from 0 to n. We consider a one-parameter Lie group of infinitesimal transformations acting on the independent and dependent variables of the system (2)

$$\begin{aligned} \tilde{x}^i &= x^i + s\xi^i(x, u) + O(s^2), \qquad & i = 1 \cdots, p, \\ \tilde{u}^j &= u^j + s\eta^j(x, u) + O(s^2), \qquad & j = 1 \cdots, q, \end{aligned} \tag{3}$$

where s is the parameter of the transformation and ξ^i, η^j are the infinitesimals of the transformations for the independent and dependent variables, respectively. The infinitesimal generator $\mathbf{v}$ associated with the above group of transformations can be written as

$$\mathbf{v} = \sum_{i=1}^{p} \xi^i(x, u)\partial_{x^i} + \sum_{j=1}^{q} \eta^j(x, u)\partial_{u^j}. \tag{4}$$

A symmetry of a differential equation is a transformation which maps solutions of the equation to other solutions. The invariance of the system (2) under the

infinitesimal transformations leads to the invariance conditions (Theorem 2.36 of [11])

$$\mathrm{Pr}^{(n)}\mathbf{v}\big[\Delta_\nu(x,u^{(n)})\big]=0, \qquad \nu=1,\cdots,l, \quad \text{whenever} \quad \Delta_\nu(x,u^{(n)})=0, \tag{5}$$

where $\mathrm{Pr}^{(n)}$ is called the n^{th} order prolongation of the infinitesimal generator given by

$$\mathrm{Pr}^{(n)}\mathbf{v}=\mathbf{v}+\sum_{\alpha=1}^{q}\sum_{J}\varphi_\alpha^J(x,u^{(n)})\partial_{u_J^\alpha}, \tag{6}$$

where $J=(j_1,\cdots,j_k)$, $1\le j_k\le p$, $1\le k\le n$ and the sum is over all J's of order $0<\#J\le n$. If $\#J=k$, the coefficient φ_J^α of $\partial_{u_J^\alpha}$ will only depend on k-th and lower order derivatives of u, and

$$\varphi_\alpha^J(x,u^{(n)})=D_J(\varphi_\alpha-\sum_{i=1}^{p}\xi^iu_i^\alpha)+\sum_{i=1}^{p}\xi^iu_{J,i}^\alpha, \tag{7}$$

where $u_i^\alpha:=\partial u^\alpha/\partial x^i$ and $u_{J,i}^\alpha:=\partial u_J^\alpha/\partial x^i$.

One of the most important properties of these infinitesimal symmetries is that they form a Lie algebra under the usual Lie bracket.

3 Lie symmetries of the equation (1)

We consider the one parameter Lie group of infinitesimal transformations on $(x^1=x,x^2=t,u^1=u)$,

$$\begin{array}{rcl}\tilde{x} &=& x+s\xi(x,t,u)+O(s^2),\\ \tilde{t} &=& x+s\eta(x,t,u)+O(s^2),\\ \tilde{u} &=& x+s\varphi(x,t,u)+O(s^2),\end{array} \tag{8}$$

where s is the group parameter and $\xi^1=\xi$, $\xi^2=\eta$ and $\varphi^1=\varphi$ are the infinitesimals of the transformations for the independent and dependent variables, respectively. The associated vector field is of the form:

$$\mathbf{v}=\xi(x,t,u)\partial_x+\eta(x,t,u)\partial_t+\varphi(x,t,u)\partial_u. \tag{9}$$

and, by (6) its third prolongation is

$$\begin{array}{rcl}\mathrm{Pr}^{(5)}\mathbf{v} &=& \mathbf{v}+\varphi^x\,\partial_{u_x}+\varphi^t\,\partial_{u_t}+\varphi^{x^2}\,\partial_{u_{x^2}}+\varphi^{xt}\,\partial_{u_{xt}}+\cdots\\ && \cdots+\varphi^{t^2}\,\partial_{u_{t^2}}+\varphi^{xt^4}\,\partial_{u_{xt^4}}+\varphi^{t^5}\,\partial_{u_{t^5}}.\end{array} \tag{10}$$

where, for instance by (7) we have

$$\begin{aligned}
\varphi^x &= D_x(\varphi - \xi\, u_x - \eta\, u_t) + \xi\, u_{x^2} + \eta\, u_{xt}, \\
\varphi^t &= D_t(\varphi - \xi\, u_x - \eta\, u_t) + \xi\, u_{xt} + \eta\, u_{t^2}, \\
&\dots \\
\varphi^{t^5} &= D_x^5(\varphi - \xi\, u_x - \eta\, u_t) + \xi\, u_{x^5 t} + \eta\, u_{t^5},
\end{aligned} \tag{11}$$

where D_x and D_t are the total derivatives with respect to x and t respectively. By (5) the vector field $\mathbf{v}$ generates a one parameter symmetry group of equation (1) if and only if

$$\begin{aligned}
&\mathrm{Pr}^{(5)}\mathbf{v}[u_t + auu_x + bu_{x^3} + cu_{x^4} + du_{x^5} - eu_{x^2}] = 0, \\
&\text{whenever} \quad u_t + auu_x + bu_{x^3} + cu_{x^4} + du_{x^5} - eu_{x^2} = 0.
\end{aligned} \tag{12}$$

The condition (12) is equivalent to

$$\begin{aligned}
&au_x\varphi + au\varphi^x + \varphi^t - e\varphi^{x^2} + b\varphi^{x^3} + c\varphi^{x^4} + d\varphi^{x^5} = 0, \\
&\text{whenever} \quad u_t + auu_x + bu_{x^3} + cu_{x^4} + du_{x^5} - eu_{x^2} = 0.
\end{aligned} \tag{13}$$

Substituting (11) into (13), and equating the coefficients of the various monomials in partial derivatives with respect to x and various power of u, we can find the determining equations for the symmetry group of the equation (1). Solving this equation, we get the following forms of the coefficient functions

$$\xi = c_2 at + c_3, \quad \eta = c_1, \quad \varphi = c_2. \tag{14}$$

where c_1, c_2 and c_3 are arbitrary constant. Thus, the Lie algebra of infinitesimal symmetry of the equation (1) is spanned bye the three vector fields:

$$\mathbf{v}_1 = \partial_x, \quad \mathbf{v}_2 = \partial_t, \quad \mathbf{v}_3 = t\,\partial_x + \frac{1}{a}\partial_u. \tag{15}$$

The commutation relations between these vector fields are given in the Table 1.

Table 1: The commutator table

$[\mathbf{v}_i, \mathbf{v}_j]$	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3$
$\mathbf{v}_1$	0	0	0
$\mathbf{v}_2$	0	0	$\mathbf{v}_1$
$\mathbf{v}_3$	0	$-\mathbf{v}_1$	0

Theorem 3.1 *The Lie algebra $\pounds_3$ spanned by v_1, v_2, v_3 is second Bianchi class type and it's solvable and Nilpotent.* [7]

To obtain the group transformation which is generated by the infinitesimal generators $\mathbf{v}_i$ for $i = 1, 2, 3$ we need to solve the three systems of first order ordinary differential equations

$$\begin{aligned} \frac{d\tilde{x}(s)}{ds} &= \xi_i(\tilde{x}(s), \tilde{t}(s), \tilde{u}(s)), \quad \tilde{x}(0) = x, \\ \frac{d\tilde{t}(s)}{ds} &= \eta_i(\tilde{x}(s), \tilde{t}(s), \tilde{u}(s)), \quad \tilde{t}(0) = t, \qquad i = 1, 2, 3 \\ \frac{d\tilde{u}(s)}{ds} &= \varphi_i(\tilde{x}(s), \tilde{t}(s), \tilde{u}(s)), \quad \tilde{u}(0) = u. \end{aligned} \tag{16}$$

Exponentiating the infinitesimal symmetries of (1), we get the one-parameter groups $G_i(s)$ generated by $\mathbf{v}_i$ for $i = 1, 2, 3$

$$\begin{aligned} G_1 : (t, x, u) &\longmapsto (x + s, t, u), \\ G_2 : (t, x, u) &\longmapsto (x, t + s, u), \\ G_3 : (t, x, u) &\longmapsto (x + ts, t, u + s/a). \end{aligned} \tag{17}$$

Consequently,

Theorem 3.2 *If $u = f(x, t)$ is a solution of (1), so are the functions*

$$\begin{aligned} G_1(s) \cdot f(x, t) &= f(x - s, t), \\ G_2(s) \cdot f(x, t) &= f(x, t - s), \\ G_3(s) \cdot f(x, t) &= f(x - ts, t) + s/a. \end{aligned} \tag{18}$$

4 Optimal system and invariant solution of (1)

In this section, we obtain the optimal system and reduced forms of the equation (1) by using symmetry group properties obtained in previous section. Since the original partial differential equation has two independent variables, then this partial differential equation transform into the ordinary differential equation after reduction.

Definition 4.1 Let G be a Lie group with Lie algebra $\mathfrak{g}$. An optimal system of $s-$parameter subgroups is a list of conjugacy inequivalent $s-$parameter subalgebras with the property that any other subgroup is conjugate to precisely one subgroup in the list. Similarly, a list of $s-$parameter subalgebras forms an optimal system if every $s-$parameter subalgebra of $\mathfrak{g}$ is equivalent to a unique member of the list under some element of the adjoint representation: $\overline{\mathfrak{h}} = \mathrm{Ad}(g(\mathfrak{h}))$.[11]

Theorem 4.2 *Let H and $\overline{H}$ be connected s-dimensional Lie subgroups of the Lie group G with corresponding Lie subalgebras $\mathfrak{h}$ and $\overline{\mathfrak{h}}$ of the Lie algebra $\mathfrak{g}$ of G. Then $\overline{H}=gHg^{-1}$ are conjugate subgroups if and only if $\overline{\mathfrak{h}} = \mathrm{Ad}(g(\mathfrak{h}))$ are conjugate subalgebras.* [11]

By theorem (4.2), the problem of finding an optimal system of subgroups is equivalent to that of finding an optimal system of subalgebras. For one-dimensional subalgebras, this classification problem is essentially the same as the problem of classifying the orbits of the adjoint representation, since each one-dimensional subalgebra is determined by nonzero vector in $\mathfrak{g}$. This problem is attacked by the naïve approach of taking a general element $\mathbf{V}$ in $\mathfrak{g}$ and subjecting it to various adjoint transformation so as to "simplify" it as much as possible. Thus we will deal with th construction of the optimal system of subalgebras of $\mathfrak{g}$.

To compute the adjoint representation, we use the Lie series

$$\mathrm{Ad}(\exp(\varepsilon \mathbf{v}_i)\mathbf{v}_j) = \mathbf{v}_j - \varepsilon[\mathbf{v}_i, \mathbf{v}_j] + \frac{\varepsilon^2}{2}[\mathbf{v}_i, [\mathbf{v}_i, \mathbf{v}_j]] - \cdots, \tag{19}$$

where $[\mathbf{v}_i, \mathbf{v}_j]$ is the commutator for the Lie algebra, ε is a parameter, and $i, j = 1, 2, 3$. Then we have the Table 2.

Table 2: Adjoint representation table of the infinitesimal generators $\mathbf{v}_i$

$\mathrm{Ad}(\exp(\varepsilon\mathbf{v}_i))\mathbf{v}_j$	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3$
$\mathbf{v}_1$	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3$
$\mathbf{v}_2$	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3 - \varepsilon\mathbf{v}_1$
$\mathbf{v}_3$	$\mathbf{v}_1$	$\mathbf{v}_2 + \varepsilon\mathbf{v}_1$	$\mathbf{v}_3$

Theorem 4.3 *An optimal system of one-dimensional Lie algebras of the equation (1) is provided by* 1) $\mathbf{v}_2$, 2) $\mathbf{v}_3 + \alpha\mathbf{v}_2$

Proof: Consider the symmetry algebra $\mathfrak{g}$ of the equation (1) whose adjoint representation was determined in table 2 and

$$\mathbf{V} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3. \tag{20}$$

is a nonzero vector field in $\mathfrak{g}$. We will simplify as many of the coefficients a_i as possible through judicious applications of adjoint maps to $\mathbf{V}$. Suppose first that $a_3 \neq 0$. Scaling $\mathbf{V}$ if necessary we can assume that $a_3 = 1$. Referring to table 2, if we act on such a $\mathbf{V}$ by $Ad(\exp(a_1\mathbf{v}_2))$, we can make the coefficient of $\mathbf{v}_1$ vanish and the vector field $\mathbf{V}$ takes the form

$$\mathbf{V}' = Ad(\exp(a_1\mathbf{v}_2))\mathbf{V} = a_2'\mathbf{v}_2 + \mathbf{v}_3. \tag{21}$$

for certain scalar a'_2. So, depending on the sign of a'_2, we can make the coefficient of $\mathbf{v}_2$ either +1, -1 or 0. In other words, every one-dimensional subalgebra generated by a $\mathbf{V}$ with $a_3 \neq 0$ is equivalent to one spanned by either $\mathbf{v}_3 + \mathbf{v}_2$, $\mathbf{v}_3 - \mathbf{v}_2$ or $\mathbf{v}_3$.

The remaining one-dimensional subalgebras are spanned by vectors of the above form with $a_3 = 0$. If $a_2 \neq 0$, we scale to make $a_2 = 1$, and then the vector field $\mathbf{V}$ takes the form

$$\mathbf{V}'' = a''_1 \mathbf{v}_1 + \mathbf{v}_2. \tag{22}$$

for certain scalar a''_1. Similarly we can vanish a''_1, so every one-dimensional subalgebra generated by a $\mathbf{V}$ with $a_3 = 0$ is equivalent to the subalgebra spanned by $\mathbf{v}_2$. □

Theorem 4.4 *An optimal system of two-dimensional Lie algebras of the equation (1) is provided by*

$$< \alpha \mathbf{v}_2 + \mathbf{v}_3, \beta \mathbf{v}_1 + \gamma \mathbf{v}_3 >$$

Symmetry group method will be applied to the (1) to be connected directly to some order differential equations. To do this, a particular linear combinations of infinitesimals are considered and their corresponding invariants are determined.

The equation (1) is expressed in the coordinates (x, t, u), so to reduce this equation is to search for its form in specific coordinates. Those coordinates will be constructed by searching for independent invariants (χ, ζ) corresponding to the infinitesimal generator. So using the chain rule, the expression of the equation in the new coordinate allows us to the reduced equation.

In what follows, we begin the reduction process of equation (1).

4.5 Galilean-Invariant Solutions. First, consider $\mathbf{v}_3 = t\,\partial_x + \frac{1}{a}\partial_u$. To determine independent invariants I, we need to solve the first partial differential equations $\mathbf{v}_i$(I)=0, that is invariants ζ and χ can be found by integrating the corresponding characteristic system, which is

$$\frac{dt}{0} = \frac{dx}{t} = \frac{a\,du}{1}. \tag{23}$$

The obtained solution are given by

$$\chi = t, \qquad \zeta = u - \frac{x}{a\,t}. \tag{24}$$

Therefore, a solution of our equation in this case is

$$u = f(x, \chi, \zeta) = \zeta + \frac{x}{a\,t}. \tag{25}$$

The derivatives of u are given in terms of ζ and χ as

$$u_x = \frac{1}{a\,t}, \quad u_{x^2} = u_{x^3} = u_{x^4} = u_{x^5} = 0, \quad u_t = \zeta_\chi - \frac{1}{a\,t^2}\,x. \tag{26}$$

Substituting (26) into the equation (1), we obtain the order ordinary differential equation

$$\zeta_\chi + \frac{1}{\chi}\,\zeta = 0. \tag{27}$$

The solution of this equation is $\zeta = \frac{c_1}{\chi}$. Consequently, we obtain that

$$u(x,t) = \frac{x + a\,c_1}{a\,t}. \tag{28}$$

4.6 Travelling wave solutions. The invariants of $\mathbf{v}_2 + c_0\,\mathbf{v}_1 = c_0\,\partial_x + \partial_t$ are $\chi = x - c_0\,t$ and $\zeta = u$ so the reduced form of equation (1) is

$$-c_0\,\zeta_\chi + a\,\zeta\,\zeta_\chi + b\,\zeta_{\chi^3} + c\,\zeta_{\chi^4} + d\,\zeta_{\chi^5} - e\,\zeta_{\chi^2} = 0. \tag{29}$$

The family of the periodic solution for Eq.(29) when $a = 1$ takes the following form (see [8])

$$\zeta = a_0 + A\,sn^4\{m\,\chi, k\} + B\,sn\{m\,\chi, k\}\,\frac{d}{d\chi}sn\{m\,\chi, k\}. \tag{30}$$

where $sn\{m\,\chi, k\}$ is Jacobi elliptic function.

4.7 The invariants of $\mathbf{v}_3 + \beta\mathbf{v}_2 = t\,\partial_x + \beta\partial_t + \frac{1}{a}\partial_u$ are $\chi = x - \frac{t^2}{2\beta}$ and $\zeta = u - \frac{t}{a\beta}$ so the reduced form of equation (1) is

$$\frac{1}{a\beta} - \frac{t}{\beta}\zeta_\chi + a\,\zeta\,\zeta_\chi + b\,\zeta_{\chi^3} + c\,\zeta_{\chi^4} + d\,\zeta_{\chi^5} - e\,\zeta_{\chi^2} = 0. \tag{31}$$

4.8 The invariants of $\mathbf{v}_2 = \partial_t$ are $\chi = x$ and $\zeta = u$ then the reduced form of equation (1) is

$$a\,\zeta\,\zeta_\chi + b\,\zeta_{\chi^3} + c\,\zeta_{\chi^4} + d\,\zeta_{\chi^5} - e\,\zeta_{\chi^2} = 0. \tag{32}$$

4.9 The invariants of $\mathbf{v}_1 = \partial_x$ are $\chi = t$ and $\zeta = u$ then the reduced form of equation (1) is $\zeta_\chi = 0$, then the solution of this equation is $u(x,t) = cte$.

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